ON THE GROUP OF SIGN (0, 3; 2, 4, ∞) AND THE FUNCTIONS BELONGING TO IT*

BY

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Introduction.

The group (Γ) which is the object of the present investigation has already been briefly discussed by HURWITZ.† His paper deals chiefly with the groups of signs (0 , 3 ; 2 , 4 , ∞) and (0 , 3 ; 2 , 6 , ∞) . ‡ He proves them "commensurable" with the modular-group (0 , 3 ; 2 , 3 , ∞) and hence derives certain algebraic relations between the simplest functions belonging to his groups on the one hand and the invariant J of the modular group on the other. He also obtains the arithmetic character of the substitutions of his groups. The group Γ is also mentioned briefly in FRICKE-KLEIN'S treatise, § where it appears as the "reproducing group" of a certain ternary quadratic form.

In Part I, we derive Γ as the monodromy group of the Riemann surface R ($p = 2$) defined by the equation

$$y^4 = (x - \kappa_1)^2 (x - \kappa_2)^2 (x - \kappa_3)(x - \kappa_4)^3,$$

i. e., as the group on the parameter z involved in the periods of two independent normal integrals of the first kind on R, generated by the movement of the branch-points about any closed paths. ||

Part II contains a rather detailed discussion of Γ by the methods applied to the

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† *Über eine Reihe neuer Functionen, etc.*, Mathematische Annalen, vol. 20 (1882), pp. 125-134.

‡ The latter has been discussed by HUTCHINSON (cf. reference below). For notation cf. FRICKE-KLEIN, *Automorphe Functionen*, vol. 1, p. 383.

§ *Automorphe Functionen*, vol. 1, p. 548.

|| This and another surface [$y^3 = (x - k_1)(x - k_2)(x - k_3)^2(x - k_4)^2$] have already received notice by BURKHARDT, *Über die Darstellung einiger Fälle der automorphen Primformen durch specielle Thetareihen*, Mathematische Annalen, vol. 42 (1893), pp. 184–214. The monodromy group of the latter surface and the corresponding automorphic functions have been discussed by HUTCHINSON, *On a class of automorphic functions*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 1–11.

modular group in FRICKE-KLEIN's treatise.* In view of the commensurability of the two groups and the similarity of their generating substitutions, we are not surprised to find many properties common to Γ and the modular group. The most noteworthy difference is perhaps the fact that, whereas the latter contains only a finite number (six) of invariant subgroups of genus zero, Γ contains a simply infinite system of such subgroups of index $2n$; the corresponding quotient-groups with regard to Γ are the dihedral groups of order $2n$. A set of generators is obtained for each and the fundamental regions are constructed. It has not been possible as yet to determine the arithmetic character of the substitutions of these subgroups; but a convenient criterion is found which completely characterizes the substitutions of each (§ 3). The finite group obtained by reducing the coefficients of all the substitutions of Γ modulo n, is briefly discussed when n is an odd prime q; two cases present themselves according as 2 is a quadratic residue, or non-residue of q (§ 4).

Part III treats of the functions of z belonging to the group Γ. RITTER's automorphic forms when regarded as functions of the ratio of the homogeneous variables give rise to the Θ-functions, which are a simple generalization of POINCARÉ's theta-fuchsian functions (§ 1). With the aid of the hyperelliptic ($p = 2$) theta-constants it is possible to construct three functions θ, ϕ, ψ, by means of which every Θ-function of Γ can be rationally expressed (§ 5). Certain theorems concerning the expression of any theta-fuchsian function of degree greater than four as a product of functions of smaller degrees are deduced (§ 2). A principal automorphic function of Γ is constructed (§ 4) and shown to be a simple function of the cross-ratio of the branch-points of the surface R. The paper closes (§ 6) with the derivation of certain infinite series. Exponential series are derived for θ, ϕ, ψ, and series similar to the Poincaré series, but more convenient, for ϕ^2, ψ, and $\phi\psi$.

PART I.

The monodromies of the Riemann Surface R defined by the equation
$$y^4 = (x-\kappa_1)^2(x-\kappa_2)^2(x-\kappa_3)(x-\kappa_4)^3.$$

1. The surface R is of genus $p = 2$. Two independent integrals of the first kind on R are

$$v_1 = \int \frac{dx}{y}, \qquad v_2 = \int \frac{dx}{y'}, \tag{1}$$

where y' is obtained from y by interchanging κ_3 and κ_4. If we choose a system of cross-cuts a_i, b_i ($i = 1, 2$) as indicated in Fig. 1, and denote the moduli of periodicity as usual by A_{ik}, B_{ik} (i, $k = 1, 2$), the table of periods takes the following form

* *Elliptische Modulfunctionen*, vol. 1, pp 163–491.

	a_1	a_2	b_1	b_2
v_1	A_{11}	$-iA_{11}$	B_{11}	$-A_{11}-iB_{11}$
v_2	A_{12}	iA_{12}	B_{12}	$-A_{12}+iB_{12}$

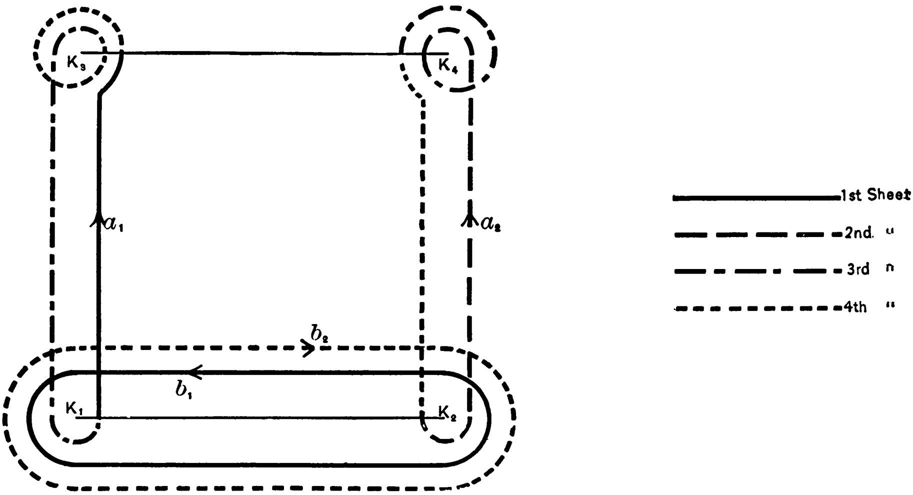

FIG. 1.

2. If we calculate the periods of two normal integrals u_1, u_2 of the first kind and if for B_{11}/A_{11} we write $z/\sqrt{2}+(1+i)/2$ $(i^2=-1)$, we may, by making use of the bilinear relation

$$A_{11}B_{12}-B_{11}A_{12}+iA_{11}A_{12}=0,$$

put the table of periods for u_1, u_2 in the form

(2)

	a_1	a_2	b_1	b_2
u_1	πi	0	$\pi i\left(\frac{z}{\sqrt{2}}+\frac{1}{2}\right)$	$-\frac{\pi i}{2}$
u_2	0	πi	$-\frac{\pi i}{2}$	$\pi i\left(\frac{z}{\sqrt{2}}+\frac{1}{2}\right)$

3. Any continuous deformation of R whereby the branch-points are caused to move in closed paths we call a monodromy of the branch-points.* Every monodromy of R can be generated by the repeated application of the following

* Cf. BURKHARDT, *Systematik der hyperelliptischen Functionen*, Mathematische Annalen, vol. 35 (1890), p. 212, for nomenclature.

two: *a*) The interchange of κ_1 and κ_2, and *b*) The movement of κ_2 about κ_3.* Denote the resulting substitutions on z by S and T_1, respectively. They are

$$S(z) = z + \sqrt{2}, \qquad T_1(z) = \frac{-\sqrt{2}z - 1}{z}.\dagger$$

Introducing in the place of T_1 the substitution $T = ST_1$, we have as generators of the group Γ,

$$S(z) = z + \sqrt{2},$$

$$T(z) = -\frac{1}{z}.$$

Part II.

The group Γ of sign $(0, 3;\ 2, 4, \infty)$.

The group Γ generated by the two substitutions S and T of the last paragraph, will now be discussed in some detail. The treatment will follow closely that of the modular group in Fricke-Klein's *Theorie der Elliptischen Modulfunctionen*, vol. I, to which reference will be made by the letter M.

§1. *The arithmetic character of the substitutions in Γ. — Fundamental region.*

1. *The group Γ consists of the totality of substitutions, V, of the two types*

$$V' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & \delta \end{vmatrix} \quad \text{and} \quad V'' = \begin{vmatrix} \alpha'\sqrt{2} & \beta' \\ \gamma' & \delta'\sqrt{2} \end{vmatrix}$$

of determinant unity, in which $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ are integers.

That every substitution of Γ is of one of the above types is easily seen. That every substitution V is in Γ is shown by proving that any V may be reduced to the identical substitution by successive multiplication by T and suitable powers of S. The reduction requires an even or an odd number of multiplications by T according as V belongs to the type V' or V''; we call the substitutions of these types even and odd respectively.

2. The following kinds of substitutions occur in Γ.

a) The even substitutions:

i) elliptic, of period 2, when $\alpha + \delta = 0$,

ii) parabolic, when $\alpha + \delta = 2$,

iii) hyperbolic, when $\alpha + \delta > 2$.

b) The odd substitutions:

i) elliptic, of period 2, when $\alpha' + \delta' = 0$,

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* Two monodromies are not regarded as distinct if they produce the same change in z.

† For a more detailed account of an analogous problem, see Hutchinson, *On a class*, etc., Transactions, vol. 3 (1902), p. 3.

ii) elliptic, of period 4, when $\alpha' + \delta = 1$,

iii) hyperbolic, when $\alpha' + \delta' > 1$.

3. A fundamental region for Γ in the $z = z' + iz''$ plane is the infinite strip included between the two straight lines $z' = \pm \frac{1}{2}\sqrt{2}$ and exterior to the circle with unit radius about the origin. The points $z = i$, $z = \rho = (-1 + i)/\sqrt{2}$, and $z = i\infty$ are fixed points of the substitutions T, $U = S^{-1}T$, and S, of which the first two are subject to the relations $T^2 = 1$, $U^4 = 1$. (See the region marked 1 in Fig. 2.)

§ 2. *Sets of conjugate substitutions in* Γ.

4. *Elliptic substitutions of period* 4 in Γ are of two classes, according as the revolution about one of their fixed points due to a single application of the substitution is $\pi/2$ or $3\pi/2$. As in M. p. 263, we show that *all substitutions of period* 4 *of the former class are conjugate with* U*; all of the latter with* U^3*; and no substitution of the former is conjugate with any substitution of the latter.*

Further, *all even substitutions of period* 2 *are conjugate with* U^2 *and all odd substitutions of period* 2 *with* T.

5. Any parabolic substitution

$$S' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & 2-\alpha \end{vmatrix}$$

may be written in the form

$$S' = \begin{vmatrix} 1 + kcd & \frac{kd^2}{2}\sqrt{2} \\ -kc^2d\sqrt{2} & 1 - kcd \end{vmatrix},$$

where c, d are prime to each other and such that $(\alpha - 1)/\gamma = -d/c$, and where k is an integer. Either k or d is even. We define the *amplitude* of S' as follows:

If d is even, the amplitude is k; if d is odd, the amplitude is $\frac{1}{2}k$.

With this definition we may show that *two parabolic substitutions of* Γ *are conjugate if, and only if, they have the same amplitude* (M. pp. 264, 265).

For, if d is even, we may determine two numbers a, b such that $ad - bc = 1$, and if we place

$$W = \begin{vmatrix} a\sqrt{2} & b \\ c & \frac{d}{2}\sqrt{2} \end{vmatrix},$$

we have

$$WS'W^{-1} = \begin{vmatrix} 1 & k\sqrt{2} \\ 0 & 1 \end{vmatrix} = S^k.$$

If d is odd we may determine two numbers a', b' such that $a'd - 2b'c = 1$, and if we write

$$W' = \begin{vmatrix} a' & b'\sqrt{2} \\ c\sqrt{2} & d \end{vmatrix},$$

we have

$$W'S'W'^{-1} = \begin{vmatrix} 1 & \frac{k}{2}\sqrt{2} \\ 0 & 1 \end{vmatrix} = S^{\frac{1}{2}k}.$$

6. No use will be made in the following of the conditions for the conjugacy of two hyperbolic substitutions. Their derivation, involving a discussion of a certain class of binary quadratic forms, may therefore be omitted.

§ 3. *The invariant subgroups of* Γ.

7. In discussing the subgroups of Γ we confine ourselves to those which are invariant (self-conjugate) and assume familiarity with the general methods employed in M. pp. 305 ff. An invariant subgroup of Γ of index μ and genus p we denote by $\Gamma_{\mu,p}$; the fundamental region of the subgroup consisting of μ fundamental triangles,—or the corresponding closed surface,—by $F_{\mu,p}$. The outer edges of the triangles correspond in pairs and the substitutions of Γ which transform two corresponding edges one into the other, form a set of generators of $\Gamma_{\mu,p}$.

The genus of an invariant subgroup $\Gamma_{\mu,p}$ is given by the equation (M. p. 341):

$$p = -\mu + 1 + \frac{\mu}{n_i}\cdot\frac{n_i - 1}{2} + \frac{\mu}{n_\rho}\cdot\frac{n_\rho - 1}{2} + \frac{\mu}{n_\infty}\cdot\frac{n_\infty - 1}{2},$$

where $2n_i$, $2n_\rho$, $2n_\infty$ are respectively the numbers of elementary triangles surrounding the points i, $\rho = (-1 + i)/\sqrt{2}$, $i\infty$, or their equivalents on $F_{\mu,p}$. Clearly, n_i is either 1 or 2, and n_ρ either 1, 2, or 4; the symbol $\{n_i, n_\rho, n_\infty\}$ we call the *characteristic* of the subgroup.

A consideration of the above equation gives the following possible solutions:

i) $p = 0$, $\mu = 0$, characteristic: $\{1, 1, 1\}$,

$\mu = 2$, " $\{2, 1, 2\}$,

$\mu = 2$, " $\{1, 2, 2\}$,

$\mu = 4$, " $\{1, 4, 4\}$,

$\mu = 2n$, " $\{2, 2, n\}$ $(n = 1, 2, 3, \cdots, \infty)$,

$\mu = 8$, " $\{2, 4, 2\}$,

$\mu = 24$, " $\{2, 4, 3\}$,

ii) $p = 1$, μ is indeterminate, characteristic $\{2, 4, 4\}$,

iii) $p > 1$, μ is determinate, " $\{2, 4, n\}$.

The first solution leads to Γ itself. For the next three it is easy to construct fundamental regions to show that corresponding subgroups exist. In all the other solutions the characteristic is either of the form $\{2, 2, n\}$ or $\{2, 4, n\}$ (n a positive integer), and for the former the genus of the corresponding subgroup, if it exists, is necessarily $p = 0$, and the index $\mu = 2n$.

8. We denote by "the division $(2, 2, n)$" the division of the entire plane into congruent triangles whose angles are $\pi/2$, $\pi/2$, π/n; this is the familiar dihedral division. Now, as in M. p. 356, we may establish a correspondence between the division $(2, 2, n)$ and the division of the plane defined by Γ. Moreover this correspondence will define an invariant subgroup of Γ for every positive integral value of n (M. pp. 344–352). Hence

Every division $(2, 2, n)$ *defines an invariant subgroup* $\Gamma_{2n,0}$ *of* Γ, *of index* $2n$ *and genus zero. The corresponding quotient-group* $\Gamma/\Gamma_{2n,0} = G_{2n}$ *is holomorphic* with the dihedral group of order* $2n$.

This establishes the existence in Γ of *an infinite series of invariant subgroups of genus zero*, a result in marked contrast to the modular group.

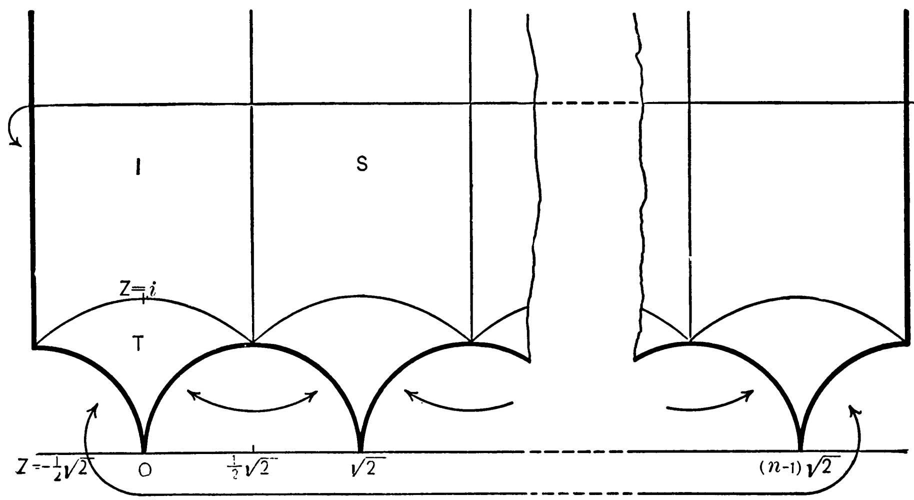

FIG. 2.

9. The accompanying figure (Fig. 2) gives a fundamental region for Γ_{2n}, which is easily obtained from the correspondence above noted. The correspondence of the edges shows that of the $n + 1$ generators thus defined, $n - 1$ are

* "Holomorphic" is used for "holoedrically isomorphic."

elliptic of period two and one is parabolic of amplitude n. In fact, the generators are at once seen to be:

$$(ST)^2,\ S(ST)^2S^{-1},\ S^2(ST)^2S^{-2},\ \cdots,\ S^{n-2}(ST)^2S^{-n+2},\ S^n,\ S^n\cdot(S^{-1}T)^2.$$

The last of these may evidently be replaced by $(S^{-1}T)^2=U^2$, so that $\Gamma_{2n,0}$ *is generated by n even substitutions of period two and one parabolic substitution of amplitude n.* Since $\Gamma_{2n,0}$ is invariant, it follows by Arts. 4, 5 that $\Gamma_{2n,0}$ *contains all the even substitutions of period two and all the parabolic substitutions of amplitude n.*

This follows also from a consideration of the correspondence established between the division $(2, 2, n)$ and the Γ-division; in fact it follows without any reference to the correspondence of sides of $F_{2n,0}$, that $\Gamma_{2n,0}$ is generated by the totality of substitutions conjugate with U^2 and S^n.

10. That all substitutions of $\Gamma_{2n,0}$ are even, is evident. But the determination of the arithmetic character of the substitutions of $\Gamma_{2n,0}$ offers as yet insurmountable difficulties of a number-theoretic character. It is however possible to give a simple criterion by which to determine whether or not a substitution is in $\Gamma_{2n,0}$.

Elementary group-theoretic considerations show that *in the isomorphism of Γ with G_{2n}, the even substitutions of Γ correspond to the cyclic subgroup H_n of G_{2n}, of order n; while the odd substitutions of Γ correspond to the remaining operations of G_{2n}.** Now any even substitution of Γ is of the form

$$V' = S^{a_1}\cdot TS^{a_2}T\cdot S^{a_3}\cdot TS^{a_4}T\cdot \cdots.$$

Since S corresponds to an operation s of period n in G_{2n}, V' will correspond to $s^{a_1-a_2+a_3-a_4+\cdots}$, which is identity only if the exponent is divisible by n. Hence *V' is a substitution of $\Gamma_{2n,0}$ if, and only if,*

$$\alpha_1-\alpha_2+\alpha_3-\alpha_4+\cdots\equiv 0 \quad (mod.\ n).$$

Any substitution V' can be written as a continued fraction:

$$V'(z)=\alpha_1\sqrt{2}+\frac{1}{-\alpha_2\sqrt{2}}\;\frac{1}{-\alpha_3\sqrt{2}}\;\frac{1}{-\alpha_4\sqrt{2}-}\cdots\frac{1}{-(\alpha_r\sqrt{2}+z)},$$

where r is odd. If the sum of the successive quotients $\alpha_i\sqrt{2}$ taken alternately with a positive and negative sign is a multiple of $n\sqrt{2}$, then and only then will V' belong to $\Gamma_{2n,0}$.

11. Similarly the division $(2, 4, n)$ defines an invariant subgroup $\Gamma_{\{n\}}$, whose index $\{n\}$ is infinite when $n>3$; for the smaller values of n, we have $\{2\}=8$ and $\{3\}=24$. The subgroup $\Gamma_{\{n\}}$ is generated by the totality of parabolic substitutions of amplitude n (M. p. 357–359); these have the form:

*The latter are all of period two.

$$S_k^n = \begin{vmatrix} 1 + ncd & \frac{1}{2}nd^2\sqrt{2} \\ -nc^2\sqrt{2} & 1 - ncd \end{vmatrix} \qquad [d \text{ even},$$

$$S_{k'}^n = \begin{vmatrix} 1 + 2ncd & nd^2\sqrt{2} \\ -2nc^2\sqrt{2} & 1 - 2ncd \end{vmatrix} \qquad [d \text{ odd},$$

where c, d take all integral values prime to each other.

That these subgroups $\Gamma_{\{n\}}$ play the same important rôle in a systematic discussion of the subgroups of Γ as do the corresponding subgroups in the modular group (M. p. 360 ff.), need hardly be mentioned. As in the latter group, so here the discussion of all the subgroups of a given "class" n reduces itself to the investigation of the quotient-group $\Gamma/\Gamma_{\{n\}} = G_{\{n\}}$. This group $G_{\{n\}}$ is itself of infinite order, if $n > 3$, but, since $\Gamma_{\{n\}}$ is an invariant subgroup of $\Gamma_{2n,0}$, it follows that $G_{\{n\}}$ is isomorphic with the dihedral group G_{2n}. This gives an insight into the structure of the groups $G_{\{n\}}$, which might be of value in a further investigation.

12. *The subgroup* $\Gamma_{\{n\}}$ *contains no elliptic substitutions;* for if it did it would contain all conjugate to U^2, and then $\Gamma_{\{n\}}$ and $\Gamma_{2n,0}$ would be identical.* Also *the invariant subgroup* $\Gamma_{\infty,0}$ *of* Γ, *generated by all the conjugates of* U^2, *contains no parabolic substitutions;* for if it contained one of amplitude n it would contain all, so that the conjugates of U^2 would generate $\Gamma_{2n,0}$; this is not possible. Now $\Gamma_{\infty,0}$ and $\Gamma_{\{n\}}$ must have substitutions in common; for otherwise every substitution of one would be commutative with every substitution of the other, which is easily shown not to be the case. These common substitutions form *an invariant subgroup of* Γ, *all of whose substitutions* (*except identity*) *are hyperbolic.*

§4. *The congruence-groups of* Γ *and the corresponding finite groups.*

13. Any subgroup of Γ, the arithmetic character of which may be completely defined by congruences, we call a congruence-group of Γ. All the substitutions of Γ which are congruent to $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ (mod. n), form an invariant subgroup $\Gamma_{\mu(n)}$ of Γ, which we call the principal congruence-group (mod. n). The index $\mu(n)$ of $\Gamma_{\mu(u)}$ is finite and is easily calculated (M. p. 395).

An even substitution can not be equivalent to an odd substitution; but it must now be noted that *even and odd substitutions may give rise to the same*

* The case $n = 1$ appears to offer an exception, since the product of the two parabolic substitutions S and TST gives an elliptic substitution. But $\Gamma_{\{n\}}$ does not exist (the characteristic (2, 4, 1) is impossible) and the totality of parabolic substitutions of amplitude one does generate $\Gamma_{2,0}$ of characteristic (2, 2, 1).

reduced (mod. n) substitution, for certain values of n. In fact the supposition that

$$V' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & \delta \end{vmatrix}$$

and

$$V'' = \begin{vmatrix} \alpha'\sqrt{2} & \beta' \\ \gamma' & \delta'\sqrt{2} \end{vmatrix}$$

give rise to the same reduced substitution leads at once to the relations

$$\alpha \equiv \sigma\alpha'\sqrt{2}. \qquad \beta\sqrt{2} \equiv \sigma\beta', \qquad \gamma\sqrt{2} \equiv \sigma\gamma', \qquad \delta \equiv \sigma\delta'\sqrt{2} \pmod{n},$$

where the factor of proportionality σ is of the form $\xi\sqrt{2}$ (ξ an integer). The relation $\sigma^2 \equiv 1 \pmod{n}$, leads to the congruence

$$2\xi^2 \equiv 1 \pmod{n}$$

and if this is possible, then to every reduced substitution correspond both even and odd substitutions of Γ. Hence, *if the congruence $2\xi^2 \equiv 1 \pmod{n}$ is possible, $\Gamma_{\mu(n)}$ contains both even and odd substitutions; if this congruence is impossible, $\Gamma_{\mu(n)}$ contains only even substitutions.*

14. The quotient-group $\Gamma/\Gamma_{\mu(n)} = G_{\mu(n)}$ is obtained by reducing all the substitutions of Γ (mod. n). In case n is an odd prime q, $G_{\mu(n)}$ is easily determined. If we transform the subgroup $\Gamma_{2,0}$ consisting of all the even substitutions of Γ by the substitution $z' = z\sqrt{2}$, we obtain the totality of substitutions $\left|\begin{smallmatrix} \alpha & \beta \\ 2\gamma & \delta \end{smallmatrix}\right|$; i. e., a subgroup of the modular group. If these substitutions be reduced (mod. q), we obtain the whole reduced modular group $G_{\frac{1}{2}q(q^2-1)}$, whose properties are well known (M. pp. 419–490). Now, since the congruence $2\xi^2 \equiv 1 \pmod{q}$ is possible only when q is of the form $8h \pm 1$, we have:

*If q is a prime of the form $8h \pm 1$, $G_{\mu(q)}$ is holomorphic with the modular $G_{\frac{1}{2}q(q^2-1)}$; if q is of the form $8h \pm 3$, $G_{\mu(q)}$ is of order $q(q^2-1)$ and contains a subgroup of index 2 holomorphic with the modular $G_{\frac{1}{2}q(q^2-1)}$.**

Part III.

The functions associated with Γ.

§ 1.—*The Θ-functions.*

Let $\Theta(z)$ be a uniform function satisfying the relation

(1) $$\Theta(z_k) = \mu_k(c_k z + d_k)^d \Theta(z),$$

* A similar result has been obtained by Fricke, for a whole series of groups, of which, however, our Γ is not a member.—Mathematische Annalen, vol. 39 (1891), p. 85.

where $z_k = (a_k z + b_k)/(c_k z + d_k)$ is a substitution with determinant unity, μ_k a constant, and d an integer. In case of an elliptic substitution, μ_k must be of the form

$$\mu_k = e^{\frac{2\pi i \nu_k}{l_k}} \quad \text{or} \quad \mu_k = e^{\frac{\pi i}{l_k}(2\nu_k+1)} \qquad (\nu_k = 0, 1, 2, \cdots, l_k - 1)$$

according as d is even or odd,* where l_k is the period of the substitution in question. Let the substitution be

$$t_k = e^{\frac{2\pi i}{l_k}} t,$$

where $t = (z - z^{(1)})/(z - z^{(2)})$ and $t_k = (z_k - z^{(1)})/(z_k - z^{(2)})$.

i) If d is even, (1) may be written

$$(2) \qquad \Theta(z_k)(z_k - z^{(2)})^d = e^{\frac{\pi i(2\nu_k - d)}{l_k}} \Theta(z)(z - z^{(2)})^d,$$

and if we put

$$\Theta(z)(z - z^{(2)})^d = \sum_{p=0}^{\infty} A_p t^p,$$

by substituting the value of t_k in terms of t, we obtain from (2) $A_p = 0$, except when $p \equiv \nu_k - \frac{1}{2}d \pmod{l_k}$. Hence, $\Theta(z)$ *vanishes at* $z = z^{(1)}$, *except when* $\nu_k - \frac{1}{2}d \equiv 0 \,(mod.\ l_k)$.

ii) Similarly if d is odd, we find $A_p = 0$, except when $p \equiv \nu_k - \frac{1}{2}(d-1) \pmod{l_k}$, and hence $\Theta(z)$ *vanishes at* $z = z^{(1)}$, *except when* $\nu_k - \frac{1}{2}(d-1) \equiv 0 \pmod{l_k}$.†

Applying these results to our group Γ, we seek first the difference between the number r of zeros and the number q of poles which lie inside the fundamental region F of Γ. We have

$$2\pi i(r - q) = \int_{(F)} d \log \Theta(z),$$

the integral being taken over the entire boundary of F, the vertices $z = \rho$, $\rho_1 (= (1+i)/\sqrt{2})$, i, $i\infty$ being excluded by small circular arcs. In evaluating this integral we make use of the relations

$$\Theta(z + \sqrt{2}) = e^{\frac{\pi i s}{4}} \Theta(z) \qquad \begin{pmatrix} s = 0, 2, 4, 6 \text{ where } d \text{ is even} \\ s = 1, 3, 5, 7 \quad \text{"} \quad d \text{ is odd} \end{pmatrix},$$

$$\Theta\left(-\frac{1}{z}\right) = e^{\frac{\pi i t}{2}} z^d \Theta(z) \qquad \begin{pmatrix} t = 0, 2 \text{ where } d \text{ is even} \\ t = 1, 3 \quad \text{"} \quad d \text{ is odd} \end{pmatrix},$$

and of the expansions

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* FRICKE-KLEIN, *Automorphe Functionen*, vol. 2, p. 85.

† For special case of the above ($\nu_k = 0$), cf. POINCARÉ, Acta Mathematica, vol. 1 (1882), pp. 218, 219.

$$\Theta(z) = \left(\frac{z-\rho}{z-\rho'}\right)^{p_0} P_1\left(\frac{z-\rho}{z-\rho'}\right), \text{ where } P_1 \text{ does not vanish when } z = \rho,$$

$$\Theta(z) = \left(\frac{z-i}{z+i}\right)^{p'_0} P_2\left(\frac{z-i}{z+i}\right), \quad \text{" } P_2 \text{ " " " " } z = i,$$

$$\Theta(z) = e^{\frac{\pi i s z}{4\sqrt{2}}} P_3\left(e^{\frac{2\pi i z}{\sqrt{2}}}\right), \quad \text{" } P_3 \text{ " " " " } z = i\infty.$$

Here p_0 and p'_0 are integers satisfying the congruences

$$p_0 \equiv t - \tfrac{1}{2}(s + d) \pmod{4},$$
$$p'_0 \equiv \tfrac{1}{2}(t - d) \pmod{2};$$

these relations are readily obtained from those on the preceding page, by observing that for p_0 we have,

$$\nu_k = t - \tfrac{1}{2}s \qquad \text{or} \qquad \nu_k = t - \tfrac{1}{2}(s+1),$$

according as d is even or odd, since the substitution in question is $S^{-1}T$; and for p'_0 we have

$$\nu_k = \tfrac{1}{2}t \qquad \text{or} \qquad \nu_k = \tfrac{1}{2}(t-1),$$

according as d is even or odd.

With these specifications the above integral may be evaluated without difficulty. We obtain, then:

The difference between the number of zeros and the number of poles of a Θ-function belonging to Γ, which lie inside the region F, is given by

$$r - q = \tfrac{1}{8}(d - 2p_0 - 4p'_0 - s).$$

§ 2. *The theta-fuchsian functions of Poincaré.*

POINCARÉ's theta-fuchsian functions* are the Θ-functions above defined for the particular values $\mu_k = 1$, i. e., $s = t = 0$. In this case d is necessarily even, equal to $2m$ say; m is called the *degree* of the function. The zeros common to all theta-fuchsian functions of a given degree, are the *fixed* zeros; the others, the *movable* zeros.

Let $\Theta^{(m)}$ denote a theta-fuchsian function of degree m, which does not become infinite anywhere in the region F; such a function we call an integral theta-fuchsian function. Now $\Theta^{(m)}$ vanishes at $z = \rho$ of order p_0 at least, where p_0 is the smallest integer satisfying the congruence $p_0 + m \equiv 0 \pmod{4}$; and at $z = i$ of order p'_0 at least, where p'_0 is the smallest integer satisfying $p'_0 + m \equiv 0 \pmod{2}$. All other zeros are movable. If $\Theta^{(m)}$ vanishes at $z = \rho$ or $z = i$ of

* Acta Mathematica, vol. 1 (1882), p. 210.

an order higher than p_0 or p_0' respectively, we shall say that one or more movable zeros have merged into the point in question. With these conventions the number of movable zeros of $\Theta^{(m)}$ is given by

$$r = \tfrac{1}{4}(m - p_0 - 2p_0'),$$

or if we place $m = 4k + k'$, then

$$r = k \qquad (k' = 0, 2, 3),$$

$$r = k - 1 \qquad (k' = 1).$$

Hence a $\Theta^{(1)}$ does not exist; a $\Theta^{(2)}$ has a single zero, of order 2, at $z = \rho$; a $\Theta^{(3)}$ has a zero of order 1 at $z = \rho$ and one of order 1 at $z = i$ and nowhere else. If we regard functions which differ only by a constant factor as not distinct, it follows that *two integral theta-fuchsian functions of the same degree and with the same zeros are identical.* For the quotient of two such functions, being an automorphic function with no zeros or infinities in F, is constant. In particular, there can exist only one $\Theta^{(2)}$ and one $\Theta^{(3)}$.

A $\Theta^{(4)}$ has a zero of order 1 at some point in F. If $\Theta_1^{(4)}$ and $\Theta_2^{(4)}$ are two functions which do not have the same zero, and if a_1 and a_2 are respectively their values at an arbitrary point $z = \alpha$ in F, then

$$\Theta_\alpha^{(4)} = a_2\Theta_1^{(4)} - a_1\Theta_2^{(4)}$$

is a $\Theta^{(4)}$ which vanishes at $z = \alpha$. Hence, since two distinct $\Theta^{(4)}$'s exist (cf. below), *a $\Theta^{(4)}$ exists with any assigned zero, and any $\Theta^{(4)}$ is linearly expressible in terms of two of them.*

Further $\Theta^{(2)}\Theta^{(3)}$ is the only $\Theta^{(5)}$. Now let $\Theta_a^{(4k)}$ be a $\Theta^{(4k)}$ with k zeros $a_i (i = 1, 2, \cdots, k)$ and let $\Theta_{a_i}^{(4)}$ be the $\Theta^{(4)}$ which vanishes at a_i. Then

$$\frac{\Theta_a^{(4k)}}{\Theta_{a_1}^{(4)} \cdot \Theta_{a_2}^{(4)} \cdots \Theta_{a_k}^{(4)}}$$

is automorphic; and, since it has no zeros or infinities in F, it is a constant. Hence:

Every integral theta-fuchsian function of degree $4k$ is expressible in one way as the product of k functions of degree 4.

Similarly, every integral theta-fuchsian function of degree $4k+\alpha$ $(\alpha=2, 3)$ is the product of $\Theta^{(\alpha)}$ and k functions of degree 4; and every function of degree $4k + 1$ is the product of $\Theta^{(2)}\Theta^{(3)}$ and $k - 1$ functions of degree 4.

§ 3. *Transformation of the theta constants.*

In order actually to construct functions of the kind we have been considering, we make use of the hyperelliptic $(p = 2)$ theta-constants,* whose moduli are

* i. e., theta-functions with zero arguments considered as functions of the parameter z entering the moduli.

given by the table (2) in Part I. These are uniform functions of $z = z' + iz''$, defined for all points such that $z'' > 0$. Our object is to determine the effect on these theta-constants when z is transformed according to the substitutions of Γ. In KRAZER and PRYM's* notation, let a_{jk} be the old moduli and b_{jk} the new. Then for the substitution S, we have

$$a_{11} = a_{22} = \pi i\left(\frac{z}{\sqrt{2}} + \tfrac{1}{2}\right),$$

$$b_{11} = b_{22} = \pi i\left(\frac{z}{\sqrt{2}} + \tfrac{3}{2}\right),$$

$$a_{12} = a_{21} = b_{12} = b_{21} = -\frac{\pi i}{2}.$$

We must determine integers α_{jk}, β_{jk}, γ_{jk}, δ_{jk} satisfying the relations

$$\sum_{\kappa} b_{\nu\kappa}\left[\alpha_{\kappa\mu}\pi i + \sum_{\lambda}\beta_{\kappa\lambda}a_{\mu\lambda}\right] = \pi i\left[\gamma_{\nu\mu}\pi i + \sum_{\lambda}\delta_{\nu\lambda}a_{\mu\lambda}\right] \quad (\kappa, \lambda, \mu, \nu = 1, 2),$$

and certain bilinear relations.† By substituting the values of α_{jk} and β_{jk}, and equating like powers of z, we easily find the most general values for α_{jk}, β_{jk}, γ_{jk}, δ_{jk}. A particular set of values is the following:

$$\alpha_{11} = \alpha_{22} = \gamma_{11} = \gamma_{22} = \delta_{11} = \delta_{22} = 1,$$

$$\beta_{jk} = \alpha_{12} = \alpha_{21} = \gamma_{12} = \gamma_{21} = \delta_{12} = \delta_{21} = 0.$$

If, then, we denote by $\vartheta\left[\begin{smallmatrix} g_1 & g_2 \\ h_1 & h_2 \end{smallmatrix}\right](z)$ a theta-function with characteristic g_i, h_i, arguments zero and moduli a_{ik}, this set of values gives the following formula:

$$\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(z + \quad \overline{2}) = e^{-\pi i(g_1^2+g_2^2-g_1-g_2)}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ g_1 + h_1 - \frac{1}{2} & g_2 + h_2 - \frac{1}{2} \end{bmatrix}(z).$$

Similarly, we obtain for the substitution T, if we place

$$b_{11} = b_{22} = \pi i\left(\tfrac{1}{2} - \frac{1}{z\sqrt{2}}\right),$$

the transformation represented by

$$\alpha_{11} = \alpha_{12} = \gamma_{11} = \delta_{11} = \delta_{21} = 0,$$

$$\alpha_{22} = \beta_{11} = \beta_{12} = \beta_{21} = \gamma_{22} = \delta_{12} = 1, \qquad \alpha_{21} = \beta_{22} = \gamma_{12} = \gamma_{21} = \delta_{22} = -1;$$

and hence the corresponding formula

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* *Neue Grundlagen*, etc.

† KRAZER und PRYM, loc. cit., p. 121: T_1 or T_2

$$\vartheta\begin{bmatrix}\hat{g}_1 & \hat{g}_2 \\ \hat{h}_1 & \hat{h}_2\end{bmatrix}\left(-\frac{1}{z}\right) = ze^{-\pi i[(g_1-g_2)^2+2g_1h_1+4g_2h_2-2g_1h_2+2h_2^2-g_1+g_2-2h_1+\frac{3}{4}]}\,\vartheta\begin{bmatrix}g_1 & g_2 \\ h_1 & h_2\end{bmatrix}(z),$$

where

$$\hat{g}_1 = -h_1 - h_2, \qquad \hat{g}_2 = g_2 - g_1 + h_2 - h_1 - 1,$$

$$\hat{h}_1 = g_2 + h_2 - \tfrac{1}{2}, \qquad \hat{h}_2 = g_1 - g_2 - h_2 - \tfrac{1}{2}.$$

Further by considering the transformation for which we have $b_{jk} = a_{jk}$, we obtain identical relations among the theta constants. The most general transformation of this kind is given by the following:

i) $\alpha_{11} = \alpha_{22} = \delta_{11} = \delta_{22} = m, \quad \alpha_{21} = \gamma_{22} = \delta_{21} = n, \quad \alpha_{12} = \gamma_{11} = \delta_{12} = -n,$

$$\beta_{jk} = \gamma_{12} = \gamma_{21} = 0;$$

ii) $\alpha_{11} = \gamma_{12} = \delta_{11} = m, \quad \alpha_{22} = \gamma_{21} = \delta_{22} = -m, \quad \alpha_{12} = \alpha_{21} = \delta_{12} = \delta_{21} = n,$

$$\beta_{jk} = \gamma_{11} = \gamma_{22} = 0;$$

where m, n are integers satisfying the relation $m^2 + n^2 = 1$. By considering all possible cases of these transformations and confining our attention to half-integer characteristics, we obtain a series of identities, which are all deducible from the two following:

$$\vartheta\begin{bmatrix}g_1 & g_2 \\ h_1 & h_2\end{bmatrix}(z) = \vartheta\begin{bmatrix}g_2 & g_1 \\ h_2 & h_1\end{bmatrix}(z),$$

$$\vartheta\begin{bmatrix}g_1 & g_2 \\ h_1 & h_2\end{bmatrix}(z) = e^{2g_1g_2\pi i}\,\vartheta\begin{bmatrix}g_1 & g_2 \\ h_1 - g_2, & g_1 - h_2\end{bmatrix}(z) \qquad (g_i,\ h_i \text{ half-integers}).$$

These relations reduce the ten theta-constants with half-integer characteristics to five, which we designate as follows: *

$$\theta = \vartheta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z),$$

$$\theta_1 = \vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z), \qquad \theta_3 = \vartheta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](z),$$

$$\theta_2 = \vartheta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right](z), \qquad \theta_4 = \vartheta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z) = -i\vartheta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](z).$$

From the general relations among the theta constants we obtain a set of identities, which may all be deduced from three, viz.:

(3) $$\theta_3^2 + \theta_4^2 = \theta_1^2, \qquad \theta_2^2 + \theta_4^2 = \theta_3^2, \qquad \theta_3^2\theta_4^2 = \theta^4.$$

From the formulæ obtained above we may exhibit the effects of the substitutions S, T on $\theta, \theta_1, \theta_2, \theta_3, \theta_4$ in the following table:

* The denominators in the characteristics have been omitted.

	θ	θ_1	θ_2	θ_3	θ_4
S	$e^{\pi i/4}\theta$	θ_2	θ_1	θ_3	$i\theta_4$
T	$iz\theta$	$e^{-\pi i/4}z\theta_2$	$e^{-3\pi i/4}z\theta_1$	$e^{-3\pi i/4}z\theta_4$	$e^{-\pi i/4}z\theta_3$

We note at once that $\theta_1^2\theta_2^2$ is a theta-fuchsian function of degree 2, which accordingly vanishes at $z=\rho$ and at no other point of F. * It is the $\Theta^{(2)}$ of the preceding article. Also θ^8 is a $\Theta^{(4)}$ which vanishes at $z=i\infty$. We have thus in $\theta_1^4\theta_2^4$ and θ^8, two linearly independent $\Theta^{(4)}$'s.

§ 4. — *A principal automorphic function of* Γ.

The function

$$\zeta=\frac{\theta_1^4\theta_2^4}{\theta^8}$$

is an automorphic function which has a single zero and a single infinity in F; it is, therefore, a *principal* function.

Every automorphic function of Γ *is rationally expressible in terms of* ζ.†

The Rosenhain moduli are ‡

$$\kappa^2=\frac{\theta_4^2}{\theta_1^2},\qquad \lambda^2=\frac{\theta_4^2}{\theta_3^2},\qquad \mu^2=\frac{\theta^4}{\theta_1^2\theta_3^2}.$$

Hence from (3) we have

(4) $$\mu^2=2\kappa^2,\qquad \frac{1}{\kappa^2}-\frac{1}{\lambda^2}=1.$$

We express the Rosenhain moduli in terms of the branch points $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ of R (Part I) by transforming the integrals (1) into the Rosenhain normal form by means of a birational transformation. The following will serve:

$$\xi=\sqrt{\frac{x-\kappa_3}{x-\kappa_4}}=\frac{y^2}{(x-\kappa_1)(x-\kappa_2)(x-\kappa_4)^2}.$$

We readily obtain

$$\frac{2}{\kappa_4-\kappa_3}\cdot\frac{(x-\kappa_4)^{\frac{5}{4}}d\xi}{(x-\kappa_3)^{\frac{1}{4}}(x-\kappa_1)^{\frac{1}{2}}(x-\kappa_2)^{\frac{1}{2}}}=\frac{dx}{y'}.$$

If we place

$$\frac{(x-\kappa_1)^{\frac{1}{2}}(x-\kappa_2)^{\frac{1}{2}}(x-\kappa_3)^{\frac{1}{4}}}{(x-\kappa_4)^{\frac{5}{4}}}=\eta,$$

* In fact, θ_1 vanishes at $z=e^{\pi i/4}$ and θ_2 at $z=\rho=e^{3\pi i/4}$.

† FRICKE-KLEIN, loc. cit., vol. 2, pp. 19, 20.

‡ KRAUSE, *Transformation Hyperelliptischer Functionen*, p. 36.

we have

$$v_1 = \frac{2}{\kappa_4 - \kappa_3}\int \frac{\xi d\xi}{\eta}, \qquad v_2 = \frac{2}{\kappa_4 - \kappa_3}\int \frac{d\xi}{\eta}.$$

Calculating η in terms of ξ, we have

$$(\kappa_4 - \kappa_3)^2\eta^2 = \xi\,[(\kappa_4 - \kappa_1)\xi^2 - (\kappa_3 - \kappa_1)]\,[(\kappa_4 - \kappa_2)\xi^2 - (\kappa_3 - \kappa_2)].$$

The branch-points of $\eta(\xi)$ are given by the roots of $\eta = 0$; if we place

$$\sqrt{\frac{\kappa_3 - \kappa_1}{\kappa_4 - \kappa_1}} = M \qquad \text{and} \qquad \sqrt{\frac{\kappa_3 - \kappa_2}{\kappa_3 - \kappa_2}} = N,$$

they are

$$\xi_1 = 0, \qquad \xi_2 = M, \qquad \xi_3 = N, \qquad \xi_4 = \infty, \qquad \xi_5 = -M, \qquad \xi_6 = -N.$$

Then, we have,

$$\kappa^2 = \frac{\xi_1 - \xi_5}{\xi_2 - \xi_5} : \frac{\xi_1 - \xi_3}{\xi_2 - \xi_3} = (\xi_1\xi_5\xi_2\xi_3) = \frac{N - M}{2N},$$

$$\lambda^2 = (\xi_1\xi_6\xi_2\xi_3) = \frac{N - M}{N + M},$$

$$\mu^2 = (\xi_1\xi_4\xi_2\xi_3) = \frac{N - M}{N}.$$

These satisfy the relations (4).

Now, we have

$$\zeta(z) = \tfrac{1}{4}\frac{(1 - 2\kappa^2)^2}{\kappa^4(1 - \kappa^2)^2} = 4\frac{\left(\frac{M}{N}\right)^2}{\left(1 - \left(\frac{M}{N}\right)^2\right)^2},$$

and

$$\left(\frac{M}{N}\right)^2 = (\kappa_3\kappa_1\kappa_4\kappa_2) = \sigma \text{ (say)}.$$

Hence we obtain,

$$\zeta(z) = \frac{4\sigma}{(1 - \sigma)^2},$$

and therefore:

Every automorphic function of Γ *is rationally expressible in terms of the cross-ratio* σ *of the branch-points* $\kappa_1, \kappa_2, \kappa_3, \kappa_4$; *and every rational function of* $\sigma + 1/\sigma$ *is automorphic.*

§ 5. *The* Θ*-functions of* Γ.

We denote the function $\theta_1\theta_2$ by ϕ, and define a new function ψ by the relation

$$\psi = \frac{\Theta^{(3)}}{\phi}.$$

The function ψ has a single simple zero at $z = i$. In ψ, ϕ, θ, then, we have three functions which vanish at i, ρ, $i\infty$ respectively, and which have no other zeros in F. If z be transformed according to any substitution in Γ, they remain unchanged except for certain factors; the latter are given in the following table:

	θ	ϕ	ψ
S	$e^{\pi i/4}$	1	1
T	iz	$-z^2$	$-z^4$

If the arbitrary constant multiplier be so chosen that $\psi(i\infty) = 1$, then ψ is given by the following equation:

$$\psi = \tfrac{1}{2}(\theta_1^4 + \theta_2^4).$$

Moreover from the relations (3) follows at once the identity

$$\theta^8 + \phi^4 - \psi^2 = 0.*$$

By means of ϕ and ψ all theta-fuchsian functions of Γ may be rationally expressed since ϕ^2 is $\Theta^{(2)}$ and $\phi\psi$ is $\Theta^{(3)}$, and since ϕ^4 and ψ^2 are two linearly independent $\Theta^{(4)}$'s. But further we may prove that

Any Θ-function belonging to Γ of whatever dimension or multiplier-system is a rational function of θ, ϕ, ψ.

The proof is simple; the following is a typical part. Denote by $\Theta_{s,t}$ a Θ-function whose multipliers are given by s, t (cf. Part. III, § 1). *The case* $s = 2$, $t = 0$: Here belong $\theta^2\phi$ and $\theta^2\psi$. Moreover, since here we have $p_0 \equiv -1 - m \pmod{4}$ and $p_0' \equiv -m \pmod{2}$ $(2m = d)$, every $\Theta_{2,0}$ vanishes either at $z = \rho$ or $z = i$; also it is 0^2 at $z = i\infty$. Hence either $\Theta_{2,0}/\theta^2\phi$ or $\Theta_{2,0}/\theta^2\psi$ are theta-fuchsian. The treatment of all other possible combinations s, t is similar. We find then:

Every integral Θ-function belonging to Γ is of the form $\theta^\alpha\phi^\beta\psi^\gamma\Pi_i\Theta_i^{(4)}$ $(\alpha < 8, \beta < 2, \gamma < 2)$, the $\Theta_i^{(4)}$'s being integral theta-fuchsian functions of degree 4.

The simplest Θ-functions for each multiplier-system are given by the following table:

	$s=0$	$s=2$	$s=4$	$s=6$		$s=1$	$s=3$	$s=5$	$s=7$
$t=0$	θ^8, ϕ^2, ψ^2	$\theta^2\phi$, $\theta^2\psi$	θ^4	$\theta^6\phi$, $\theta^6\psi$	$t=1$	θ	$\theta^3\phi$, $\theta^3\psi$	θ^5	$\theta^7\phi$, $\theta^7\psi$
$t=2$	ϕ, ψ	θ^2	$\theta^4\phi$, $\theta^4\psi$	θ^6	$t=3$	$\theta\phi$, $\theta\psi$	θ^3	$\theta^5\phi$, $\theta^5\psi$	θ^7

By analogy with the functions θ, ϕ, it might be supposed that ψ could be represented as the product of four theta-constants $(p = 2)$. It is perhaps of

* cf. FRICKE-KLEIN, loc. cit., vol. 2, p. 75.

sufficient interest to show that this is impossible. In fact, if the characteristic of a constituent theta-constant be $\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}$, it follows (by considering the theta-series) from the fact that ψ does not vanish at $z = i\infty$ that g_1, g_2 must be integers; and since T must transform each constituent into one of the same kind (i. e., with g_1, g_2 integers) it follows from the formula in Part III, § 3, that $h_1 + h_2$ and $h_1 - h_2$ must be integers. Hence, the constituent theta-constants in any such product would have to have half-integer characteristics. But these have already been fully discussed in regard to their behavior toward S and T, and no combination of them into products will give ψ.

§ 6. *Infinite series for* θ, ϕ, ψ.

1) *Exponential series.*

The exponential series for $\vartheta\begin{bmatrix} 10 \\ 00 \end{bmatrix}(z)$ readily takes the form

$$\theta = e^{\frac{\pi i}{8}} \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} e^{(n_1-n_2)(n_1-n_2+1)\frac{\pi i}{2}} e^{[(2n_1+1)^2+4n_2^2]\frac{\pi i z}{4\sqrt{2}}}.$$

The first factor after the signs of summation equals $+1$ or -1 according as $n_1 - n_2 \equiv {0 \atop 3}$ or $n_1 - n_2 \equiv {1 \atop 2}$ (mod. 4). The second factor is such that if (a, b) are any two values of (n_1, n_2), then the pairs $(a, \pm b)$ and $(-a-1, \pm b)$ give it the same values. By considering the numbers $n_1 - n_2$ in these four cases, it follows that all terms due to an odd n_2 cancel in pairs. We may then write the series for θ as follows:

$$e^{\frac{-\pi i}{8}} \theta = 2\sum_{n_1=0}^{\infty} \pm e^{(2n_1+1)^2\frac{\pi i z}{4\sqrt{2}}} + 4\sum_{n_1=0}^{\infty} \sum_{n_2=2,4,6,\dots}^{\infty} \pm e^{[(2n_1+1)^2+4n_2^2]\frac{\pi i z}{4\sqrt{2}}}$$

where the upper sign is used whenever $n_1 - n_2 \equiv {0 \atop 3}$ (mod. 4), the lower whenever $n_1 - n_2 \equiv {1 \atop 2}$ (mod. 4). This gives

$$e^{\frac{-\pi i}{8}} \theta = 2e^{\frac{\pi i z}{4\sqrt{2}}} - 2e^{\frac{9\pi i z}{4\sqrt{2}}} - 4e^{\frac{17\pi i z}{4\sqrt{2}}} - 6e^{\frac{25\pi i z}{4\sqrt{2}}} - \cdots.$$

To obtain a series for $\phi = \theta_1\theta_2$, we proceed as follows. The series for $\theta_1 = \vartheta\begin{vmatrix} 00 \\ 00 \end{vmatrix}(z)$ gives readily

$$(5) \qquad \theta_1 = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} i^{\epsilon_m} e^{(m_1^2+m_2^2)\frac{\pi i z}{\sqrt{2}}} \quad \begin{pmatrix} \epsilon_m = 0, \text{ when } m_1 - m_2 \text{ is even} \\ \epsilon_m = 1, \text{ when } m_1 - m_2 \text{ is odd} \end{pmatrix},$$

and since $\theta_2(z) = \theta_1(z + \sqrt{2})$, we have

$$(6) \qquad \theta_2 = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} i^{-\epsilon_n} e^{(n_1^2+n_2^2)\frac{\pi i z}{\sqrt{2}}} \quad \begin{pmatrix} \epsilon_n = 0, \text{ when } n_1 - n_2 \text{ is even} \\ \epsilon_n = 1, \text{ when } n_1 - n_2 \text{ is odd} \end{pmatrix}.$$

Multiplying (5) and (6), we have

$$\phi = \sum_{-\infty}^{\infty}{}^{4}\, i^{\epsilon_m-\epsilon_n} e^{(m_1^2+m_2^2+n_1^2+n_2^2)\frac{\pi i z}{\sqrt{2}}},$$

where m_1, m_2, n_1, n_2 take independently all integral (and zero) values from $-\infty$ to $+\infty$. Since every integer may be represented as the sum of four squares, this series may be written in the form

$$\phi = \sum_{k=0}^{\infty} A_k e^{\frac{k\pi iz}{\sqrt{2}}}, \qquad A_k = \sum_k i^{\epsilon_m - \epsilon_n},$$

where the latter summation extends over all the representations of k as the sum of four squares.

Since $\phi(z + \sqrt{2}) = \phi(z)$ it follows that A_k vanishes when k is odd; an even integer k can be represented as the sum of the squares of four integers m_1, m_2, n_1, n_2 only when $m_1 - m_2$ and $n_1 - n_2$ are either both even or both odd, so that in any case $\epsilon_m - \epsilon_n = 0$. Hence when k is even, A_k is equal to the total number of representations of k as the sum of four squares, i. e., A_k is equal to 24 times the sum of all odd divisors of k.* If k be written in the form

$$k = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots,$$

the sum of all odd divisors is

$$s(k) = \prod_i \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

With this notation, we have then

$$\phi = 1 + 24 \sum_k s(k) e^{\frac{k\pi iz}{\sqrt{2}}} \qquad (k = 2, 4, 6, \cdots, \infty).$$

This gives

$$\phi = 1 + 24 e^{\frac{2\pi iz}{\sqrt{2}}} + 24 e^{\frac{4\pi iz}{\sqrt{2}}} + 96 e^{\frac{6\pi iz}{\sqrt{2}}} + \cdots.$$

A series for $\psi = \frac{1}{2}(\theta_1^2 + i\theta_2^2)(\theta_1^2 - i\theta_2^2)$ may be obtained in a similar manner. We have

$$\theta_1^2 + i\theta_2^2 = \sum{}^4 (i^{\epsilon_m + \epsilon_n} + i^{3\epsilon_m + 3\epsilon_n + 1}) e^{(m_1^2 + m_2^2 + n_1^2 + n_2^2)\frac{\pi iz}{\sqrt{2}}}$$

$$= \sum B_k e^{\frac{k\pi iz}{\sqrt{2}}},$$

suppose, where ϵ_m, ϵ_n have the same significance as before. Let $t(k)$ be the number of representations of k as the sum of four squares. If k is odd, one of the numbers ϵ_m, ϵ_n is zero and the other unity; hence we have

$$B_k = (1 + i) t(k) \qquad (k \text{ odd}).$$

If k is even we distinguish two cases:

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* EISENSTEIN, Crelle's Journal, vol. 35 (1847), p. 133; BACHMAN, *Zahlentheorie*, vol. 4, part 1, p. 603.

1) if $k \equiv 0 \pmod{4}$, we have $\epsilon_m = \epsilon_n = 0$, and

$$B_k = (1+i)t(k) \qquad (k \equiv 0 \pmod{4}).$$

2) if $k \equiv 2 \pmod{4}$, of the numbers m_1, m_2, n_1, n_2, two are even and two are odd. Now it is easy to show that in this case one-third of the total number of representations of k give $\epsilon_m + \epsilon_n = 0$, and the remaining two-thirds give $\epsilon_m + \epsilon_n = 2$. For each of the former we obtain a coefficient $1 + i$; for each of the latter a coefficient $-1 - i$. Hence we have here

$$B_k = -\tfrac{1}{3}(1+i)t(k) \qquad (k \equiv 2 \pmod{4}).$$

In general then, we have

$$B_k = \mu_k(1+i)t(k),$$

where

$$\begin{aligned} \mu_k &= 1 && \text{if } k \text{ is odd,} \\ &= 1 && \text{if } k \equiv 0 \pmod{4}, \\ &= -\tfrac{1}{3} && \text{if } k \equiv 2 \pmod{4}. \end{aligned}$$

Similar reasoning gives

$$\theta_1^2 - i\theta_2^2 = \sum B'_k e^{\frac{k\pi i z}{\sqrt{2}}},$$

where

$$B'_k = \nu_k(1-i)t(k),$$

in which

$$\begin{aligned} \nu_k &= -1 && \text{if } k \text{ is odd,} \\ &= +1 && \text{if } k \equiv 0 \pmod{4}, \\ &= -\tfrac{1}{3} && \text{if } k \equiv 2 \pmod{4}. \end{aligned}$$

Hence, finally, since $(1+i)(1-i) = 2$ we have

$$\psi = \sum_{k=0}^{\infty} C_k e^{\frac{k\pi i z}{\sqrt{2}}},$$

where

$$C_k = \sum_{s=0}^{k} \mu_s \nu_{k-s} t(s) t(k-s).$$

Transformation by S shows that $C_k = 0$, when k is odd. Hence we need consider only even values of k. The above formula gives readily

$$C_0 = 1,\ C_2 = -80,\ C_4 = -400,\ C_6 = -2240,\ \cdots.$$

2) *Series similar to Poincaré's.*

The series, derived by POINCARÉ * to represent his theta-fuchsian functions

* Acta Mathematica, loc. cit.

and subsequently generalized by RITTER * to apply to any Θ-function with whatever multiplier-system, are well known and may be used to represent the Θ-functions of Γ, provided the dimension is not less than four. However, the possibility of their identical vanishing is troublesome. This difficulty may be avoided by forming—in a way to be immediately set forth—series similar to the POINCARÉ-RITTER series, but which contain only a portion of the terms of the latter.

Let

$$z_j = f_j(z) \qquad (j = 0, 1, 2, \cdots, \infty)$$

represent the totality of substitutions of Γ which transform the fundamental region F into a region lying between the straight lines $z' = \pm 1/\sqrt{2}\ (z = z' + iz'')$, and let

$$z_j = \frac{\alpha_j z + \beta_j \sqrt{2}}{\gamma_j \sqrt{2} z + \delta_j} \qquad \text{or} \qquad z_j = \frac{\alpha'_j \sqrt{2} z + \beta'_j}{\gamma'_j z + \delta'_j \sqrt{2}},$$

according as the f_j is even or odd. Then, *of the different denominators of all the substitutions of* Γ *each occurs once and but once as a denominator of an* f_j. That every denominator occurs is seen from the fact that, if V_k is any substitution of Γ, $S^n V_k$ is a substitution with the same denominator; and n may always be so chosen that $S^n V_k$ is equal to an f_j. To show that each denominator occurs only once, observe that if two substitutions f_j with the same denominator existed they would both transform the point i into a point whose real part in absolute value is less than $1/\sqrt{2}$. If one of these, say f_k, be even, it transforms i into

$$f_k(i) = \frac{\alpha_k i + \beta_k \sqrt{2}}{\gamma_k \sqrt{2} i + \delta_k},$$

of which the real part is

$$\left| \frac{\alpha_k \gamma_k + \beta_k \delta_k}{2\gamma_k^2 + \delta_k^2} \sqrt{2} \right| < \frac{1}{\sqrt{2}},$$

so that

$$|\alpha_k \gamma_k + \beta_k \delta_k| < \tfrac{1}{2}(2\gamma_k^2 + \delta_k^2).$$

Since the determinant of every substitution is unity, any other substitution f_l with the same denominator as f_k will have

$$\alpha_l = \alpha_k + 2\gamma_k t, \qquad \beta_l = \beta_k + \delta_k t \qquad (t \text{ an integer})$$

such that

$$|\alpha_l \lambda_k + \beta_l \delta_k| = |\alpha_k \gamma_k + \beta_k \delta_k + (2\gamma_k^2 + \delta_k^2) t| < \tfrac{1}{2}(2\gamma_k^2 + \delta_k^2);$$

* *Die eindeutigen automorphen Formen, etc.*, Mathematische Annalen, vol. 41 (1892), p. 1; cf. also FRICKE-KLEIN, loc. cit., vol. 2, pp. 137 ff.

this is impossible except when $t = 0$, or when $f_l = f_k$. Similar reasoning will apply to the case when f_k is odd and leads to the same result.

We have then without difficulty:

$$(7) \qquad \sum_{j=0}^{\infty} \left[\frac{df_j(z)}{dz} \right]^m = \sum_k \frac{1}{(\gamma_k \sqrt{2} z + \delta_k)^{2m}} + \sum_k \frac{1}{(\gamma'_k z + \delta'_k \sqrt{2})^{2m}},$$

where γ_k, δ_k and γ'_k, δ'_k run through all *different* pairs of values occurring in the denominators of the substitutions of Γ. Moreover by comparison with the well-known series *

$$\sum \frac{1}{(mz+n)^\nu} \qquad (m, n \text{ integers}),$$

we see that *the series* (7) *converges uniformly and unconditionally for all values of z which are not purely real, provided only $m > 1$.*

Now, *the function*

$$\Theta^{(m)} = \sum_j \left[\frac{df_j(z)}{dz} \right]^m$$

is theta-fuchsian of degree m.† For if $z_k = V_k(z)$ be any substitution of Γ, then we have

$$\Theta^{(m)}(z_k) = \sum_j \left[\frac{df_j(z_k)}{dz_k} \right]^m = \left(\frac{dz_k}{dz} \right)^{-m} \sum_j \left[\frac{df_j[V_k(z)]}{dz} \right]^m.$$

Now the series of denominators of $f_j V_k$ ($j = 0, 1, 2, \cdots$) is the same as the series of denominators of f_j (except possibly as regards the sign; and this difference is destroyed by the *even* exponent which affects all the denominators in the series). For, (1) all the denominators of the series f_j occur in $f_j V_k$, since if f_n be any f_j, S^a may be so chosen that $S^a f_n V_k^{-1} = f_{n'}$; and $f_{n'} V_k$ has the same denominator as f_n; and (2) no denominator occurs more than once, since if $f_n V_k$ and $f_p V_k$ had the same denominator, multiplication by certain powers of S would have to reduce them both to the same f_j; this would lead at once to a relation $S^b f_n = f_p$, which is impossible. Hence we have

$$\sum_j \left[\frac{df_j[V_k(z)]}{dz} \right]^m = \sum_j \left[\frac{df_j(z)}{dz} \right],$$

and, therefore, *the $\Theta^{(m)}(z)$, defined above, has the property*

$$\Theta^{(m)}(z_k) = \left(\frac{dz_k}{dz} \right)^{-m} \Theta^{(m)}(z),$$

where $z_k = V_k(z)$ is any substitution of Γ.

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* EISENSTEIN, Crelle, vol. 35 (1847), p. 161.

† The use of such a series is suggested in FRICKE-KLEIN, loc. cit., vol. 2, p. 156.

Moreover, *the series* (7) *does not vanish identically*, for it evidently does not vanish for $z = i\infty$. *We have then, if we write*

$$\sum \frac{1}{(\gamma_k \sqrt{2} z + \delta_k)^{2m}} = \Sigma'_{2m}, \qquad \sum \frac{1}{(\gamma'_k z + \delta'_k \sqrt{2})^{2m}} = \Sigma''_{2m},$$

$$\Sigma'_{2m} + \Sigma''_{2m} = \Sigma_{2m},$$

the following expansions:

$$\Theta^{(2)} = \phi^2 = \Sigma_4,$$

$$\Theta^{(3)} = \phi\psi = \Sigma_6,$$

$$\psi = \Sigma'_4 - \Sigma''_4.$$

CORNELL UNIVERSITY,
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